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Temporal Forcing of Wave Patterns

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The effect of temporal modulations on wave patterns induced by spatiotemporal Hopf bifurcations is discussed in the framework of amplitude equations of the Ginzburg–Landau type. The approach is well adapted to the study of pattern formation in liquid crystals which, on the other hand, provide a class of easily forced systems. A few examples, related to experimental realizations, are presented. In particular, it is shown how pure temporal modulations may stabilize standing waves or two-dimensional wave patterns in regimes where they are otherwise unstable. The properties of the defects which are associated to these structures are also discussed.

KEY WORDS: Spatiotemporal patterns; external forcing; strong resonances; waves; Hopf bifurcation.

1. INTRODUCTION

The study of the effect of external modulations on pattern-forming instabilities has recently triggered an increasing interest. For example, in the Lowe–Gollub experiment,[1] a spatial modulation of the electrohydrodynamic instability of nematics induces discommensurations and a transition toward structures with incommensurate wavelengthes. The theoretical aspects of this problem have been discussed in the framework of amplitude equations and phase dynamics.[2,3] In the case of Rayleigh–Bénard convection, an imposed flow field leads to a spatial variation of the wavelength of the patterns,[4] while in chemically active media, such a flow is able to disorganize spiral waves and leads to spatiotemporal disorder.[5] It has also been shown that these effects may be interpreted within the phase dynamical analysis of the structures.[6,7] Furthermore, Brand[6] recently showed how flow fields can affect the phase dynamics and

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proposed minimal model equations able to describe qualitatively several experimental observations. In two-dimensional systems, the selection and stability of a steady pattern may also be strongly affected by externally imposed modulations as discussed by Pismen.[8] Temporal modulations of steady spatial patterns usually lead to a mere shift of the bifurcation. However, in Rayleigh–Bénard convection, a temporal modulation of the temperature field imposed at the bottom of the fluid layer breaks the vertical symmetry around the midlayer plane and may induce the formation of hexagonal structures.[9]

In the case of spatiotemporal Hopf bifurcations, original effects occur as a result of the nonvariational character of the dynamics. In particular, unstable standing waves or two-dimensional wave patterns may be stabilized by pure spatial or temporal modulations of suitable wavelengths or frequencies[10–13] and I will discuss here some examples of the influence of such temporal forcings on the selection and stability of wave patterns and also on the properties of wave defects.

2. TEMPORAL FORCING OF 1D TRAVELING WAVES

As shown recently, pure temporal modulations imposed on Hopf bifurcations leading to wave patterns modify the selection and stability properties of these spatiotemporal structures.[10–12] For example, it was shown that, in one-dimensional systems, beyond spatiotemporal Hopf bifurcations, left and right traveling waves are linearly coupled by uniform oscillations or stady spatial modulations provided the frequency of the oscillations or the wavenumber of the modulations are close to two times the critical ones. Hence, a spatially uniform forcing may restore the left–right symmetry and transform traveling waves into standing waves in regimes where they are otherwise unstable.

Close to a bifurcation, this effect may be discussed in the framework of amplitude equations. Let us consider an isotropic system, invariant under space and time translations and parity transformations, which undergoes a Hopf bifurcation with finite wavenumber q_c and frequency ω_c. The growth rate of the unstable modes is written as

$$w(\mathbf{q}) = \varepsilon - \xi_0^2(q^2 - q_c^2)^2 + i[\omega_c + \omega_1(q^2 - q_c^2) + \omega_2(q^2 - q_c^2)^2] + O((q^2 - q_c^2)^3) \qquad (1)$$

On writing the order parameter-like variable as

$$\sigma(x, y, t) = \mathrm{Re}[A(x, y, t)\exp(\mathbf{qr} - \omega_c t) + B(x, y, t)\exp i(\mathbf{qr} + \omega_c t)], \qquad |\mathbf{q}| = q_c \qquad (2)$$

the slowly-varying envelopes of right and left traveling waves propagating in, say, the x direction satisfy, after appropriate scaling, the following amplitude equations[14–16]:

$$\begin{aligned}\dot{A} + c\nabla_x A &= \varepsilon A + (1+i\alpha)\left(\nabla_x - \frac{i\nabla_y^2}{2q_c}\right)^2 A + i\eta(\nabla_x^2 + \chi\nabla_y^2)A - (1+i\beta)A\,|A|^2 \\ &\quad - (\gamma + i\delta)A\,|B|^2 \\ \dot{B} - c\nabla_x B &= \varepsilon B + (1-i\alpha)\left(\nabla_x + \frac{i\nabla_y^2}{2q_c}\pi^{\Delta}\right. B - i\eta(\nabla_x^2 + \chi\nabla_y^2)B - (1-i\beta)B\,|B|^2 \\ &\quad - (\gamma - i\delta)B\,|A|^2\end{aligned} \tag{3}$$

When $\gamma > 1$, only traveling waves are stable, whiled for $\gamma < 1$ standing waves are the only stable structures. I will consider in the following the case $\gamma > 1$. This description is valid, for example, for convection in binary fluids. Traveling waves have also been observed in liquid crystal instabilities such as the electrohydrodynamic instability of nematics.[17] In this case, one has to take into account the intrinsic anisotropy of the system in the derivation of the corresponding amplitude equations from the underlying nematohydrodynamics. For the transitions to normal or oblique rolls, this has been done by Bodenschatz *et al.*[18] However, since such a derivation is not available yet for the transition to traveling waves, one has to rely on symmetry arguments and write the real part of the growth rate of the unstable modes as

$$\operatorname{Re}\,\omega(\mathbf{q}) = \varepsilon - \xi_0^2(q^2 - q_c^2)^2 - \rho q_y^2 - 2\eta q_x^2 q_y^2 - \tau q_y^4$$

where $0x$ is the easy axis determined by the anisotropy. Hence, the first instability occurs toward waves of wavenumber q_c traveling in the x direction, while the threshold for waves of wavenumber q and traveling in an arbitrary direction making an angle ϕ with the easy axis is given by

$$\varepsilon(\phi) = \xi_0^2(q^2 - q_c^2)^2 - \rho q^2 \sin^2\phi - q^4(2\eta \sin^2\phi \cos^2\phi + \tau \sin^4\phi)$$

In this case, the amplitude equations for the critical waves $A(x, y, t) \exp i(q_c x - \omega_c t)$ and $B(x, y, t) \exp i(q_c x + \omega_c t)]$ become

$$\begin{aligned}\dot{A} + c\nabla_x A &= \varepsilon A + (1+i\alpha)(\nabla_x^2 + \rho\nabla_y^2)A - (1-i\beta)A\,|A|^2 \\ &\quad - (\gamma + i\delta)A\,|B|^2 \\ \dot{B} - c\nabla_x B &= \varepsilon B + (1-i\alpha)(\nabla_x^2 + \rho\nabla_y^2)B - (1-i\beta)B\,|B|^2 \\ &\quad - (\gamma - i\delta)B\,|A|^2\end{aligned} \tag{4}$$

If one applies to this system a pure temporal forcing described by a field of the type $h\cos 2(\omega_c+\nu)t$, a strong resonance occurs between left and right traveling waves, and the amplitude equations become

$$\begin{aligned}\dot{A}-c\nabla_x A &= (\varepsilon+i\nu)A+(1+i\alpha)(\nabla_x^2+\rho\nabla_y^2)A-(1+i\beta)A|A|^2\\ &\quad -(\gamma+i\delta)A|B|^2+\mu B\\ \dot{B}-c\nabla_x B &= (\varepsilon-i\nu)B+(1-i\alpha)(\nabla_x^2+\rho\nabla_y^2)B-(1-i\beta)B|B|^2\\ &\quad -(\gamma-i\delta)B|A|^2+\mu A\end{aligned} \tag{5}$$

where $\mu\propto h$. As shown in refs. 10 and 11, on varying the parameters ε, ν, and μ, besides limit cycles and homoclinic orbits corresponding to modulated traveling waves, these equations may admit nontrivial fixed points (cf. Fig. 1). For $\gamma=2$ and $\delta=2\beta$, they are defined by

$$\begin{gathered} A=R\exp i\phi_A, \qquad B=R\exp i\phi_B\\ \phi_A-\phi_B=\text{cst}, \qquad \phi_A+\phi_B=\sin^{-1}\left(\frac{\nu-3\beta R^2}{\mu}\right) \\ R^2=\frac{1}{3(1+\beta^2)}\{\varepsilon+\nu\beta\pm[(\varepsilon+\nu\beta)^2-(1+\beta^2)(\varepsilon^2+\nu^2-\mu^2)]^{1/2}\}\end{gathered} \tag{6}$$

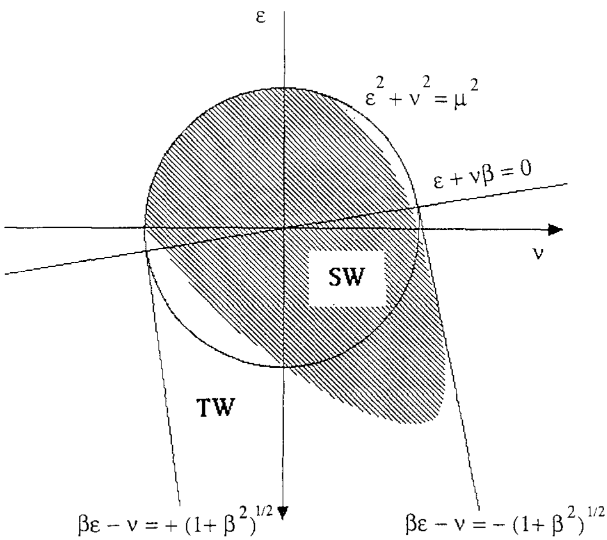


Fig. 1. Phase diagram for a Hopf bifurcation forced by temporal modulations of frequency close to twice the critical frequency, in the ε, ν plane. The two stable fixed points corresponding to standing waves are located in the hatched domain.

Only two of them are stable, as discussed in refs. 10 and 11, and the associated standing waves have been observed and analyzed in the electrohydrodynamic instability of nematics by Rehberg *et al.*[12]

The linear part of the phase dynamics associated with these forced patterns may then be written as follows [$A = R \exp i(\Phi + \psi)$, $B = R \exp i(\Phi - \psi)$, $\sigma = R \cos(qx + \Phi) \cos(\omega_c t + \psi)$]:

$$\begin{aligned} \partial_t \psi &= -c\nabla_x \Phi + D_x \nabla_x^2 \psi + D_y \nabla_y^2 \psi + \nu - (\beta + \delta) R^2 - \mu \sin 2\psi \\ \partial_t \Phi &= -c\nabla_x \psi + D_x \nabla_x^2 \Phi + D_y \nabla_y^2 \Phi \end{aligned} \tag{7}$$

Hence the standing waves correspond to the locking of the phase ψ and are expected to present excitability features. The different behavior of the phases ψ and Φ leads to different types of defects for the forced standing waves. Recall that, in the case of unforced standing waves (i.e., when $-1 < \gamma < 1$), the manifold of stable homogeneous states of Eq. (3) corresponds to a two-torus parametrized by the phases ψ and Φ, or ϕ_A and ϕ_B, and the topological defects are characterized by two topological charges (n_1, n_2) corresponding to the extra wavelengths added to the underlying right and left traveling waves.[19] The elementary defects which correspond to right $(1, 0)$ or left $(1, 0)$ dislocations have been obtained numerically[18] and experimentally.[20] In the case of forced standing waves discussed here, the manifold of stable homogeneous states is transformed into two circles parametrized by the phase Φ, since the other phase variable ψ is locked. Hence, the corresponding topological defects are (n, n) dislocations, and the most probable of them, the $(1, 1)$ dislocation, has been observed experimentally.[21]

However, as a consequence of the excitability of the ψ dynamics, defects associated with phase unlocking may also exist. For example, point defects corresponding to vertex singularities should lead to the formation of spiral waves similar to the ones observed by Lega in her analysis of defects in wave patterns.[22] On the other hand, since ψ may be locked in two different values, ψ_0 and $\psi_0 + \pi$, domains where the order parameter takes the value $R \cos(qx + \Phi) \cos(\omega_c t + \psi_0)$ or $-R \cos(qx + \Phi) \cos(\omega_c t + \psi_0)$ may coexist within the system. According to the instensity of the forcing, as discussed by Coullet *et al.*,[23] the domain walls which separate these regions may be of two types:

1. Walls where the amplitude vanishes when going from R to $-R$ as in the Ising walls of magnetic systems.

2. Walls where the amplitude remains finite, but where the phase makes $a + \pi$ or a $-\pi$ rotation as in the Bloch walls of magnetic systems. Hence, these walls are of positive or negative chirality and, as a conse-

quence of the nonvariational character of the dynamics, they move with a velocity proportional to their chirality. Furthermore, Néel points, i.e., points where the chirality changes its sign, may exist in such walls, and since the wall velocity also changes its sign at these points, they should accordingly induce the formution of spirals within the standing wave pattern.[24]

3. TEMPORAL FORCING OF 2D WAVE PATTERNS

In two-dimensional systems, new possibilities occur and I will discuss here an illustrate example. Consider the case of isotropic systems described by Eq. (3) with $\gamma > 1$, where waves traveling in one direction are the only stable structure.

The effect of a purely spatial forcing of hexagonal symmetry on such patterns has been analyzed in ref. 13. It was shown that, when the wavenumber of the forcing is close to $3q_c$, traveling waves with wavevectors parallel to the basic vectors of the imposed hexagonal modulation are nonlinearly coupled. This leads to a stabilization mechanism for two-dimensional wave patterns.

Similar effects may also be obtained with purely temporal forcings. For example, temporal modulations of frequency ω_0 such that $n\omega_0 \simeq 3\omega_c$ couple triplets of waves traveling in directions making $2\pi/3$ angles [e.g., $A_1 \exp i(\mathbf{q}_1\mathbf{r} - \omega_c t)$, $A_2 \exp i(\mathbf{q}_2\mathbf{r} - \omega_c t)$, $A_3 \exp i(\mathbf{q}_3\mathbf{r} - \omega_c t)$, with $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0$ and $|\mathbf{q}_1| = |\mathbf{q}_1| = |\mathbf{q}_1| = q_c$). The corresponding uniform amplitude equations are

$$\begin{aligned}
\dot{A}_1 &= (\varepsilon + i\nu)A_1 + v\bar{A}_2\bar{A}_3 - (1 - i\beta)A_1|A_1|^2 - (\kappa + i\lambda)A_1(|A_2|^2 + |A_3|^2) \\
\dot{A}_2 &= (\varepsilon + i\nu)A_2 + v\bar{A}_1\bar{A}_3 - (1 + i\beta)A_2|A_2|^2 - (\kappa + i\lambda)A_2(|A_1|^2 + |A_3|^2) \qquad (8) \\
\dot{A}_3 &= (\varepsilon + i\nu)A_3 + v\bar{A}_2\bar{A}_1 - (1 + i\beta)A_3|A_3|^2 - (\kappa + i\lambda)A_3(|A_1|^2 + |A_2|^2)
\end{aligned}$$

where $v \propto h^n$, h being the amplitude of the external modulation, and $\nu = n\omega_0/3 - \omega_c$, the frequency detuning between the external field and the waves.

Hence, from the fixed-point condition ($A_1 = R \exp i\Phi_1$, $A_2 = R \exp i\Phi_2$, $A_3 = R \exp i\Phi_3$):

$$[(1+2\kappa)^2 + (\beta+2\lambda)^2]R^4 - [2\varepsilon(1+2\kappa) + 2\nu(\beta+2\lambda) + v^2]R^2 + (\varepsilon^2+\nu^2) = 0 \qquad (9)$$

it may be deduced that hexagonal wave patterns exist when

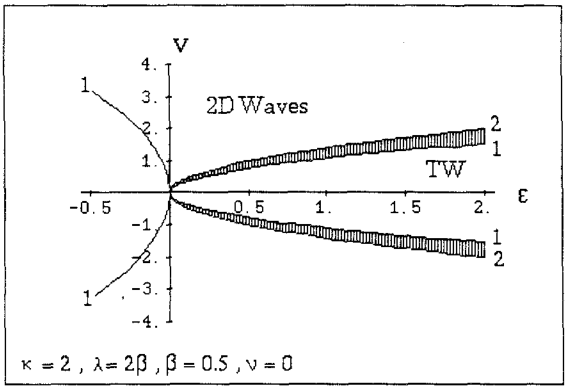
$$v^4 + 4v^2[\varepsilon(1+2\kappa) + \nu(\beta+2\lambda)] - 4[\nu(1+2\kappa) - \varepsilon(\beta+2\lambda)]^2 > 0 \qquad (10)$$

and are the selected structures when traveling waves are unstable, i.e., when

$$v^2\varepsilon - (\nu - \lambda\varepsilon)^2 - (\kappa - 1)^2\varepsilon^2 > 0 \tag{11}$$

The corresponding phase diagram is presented in Fig. 2 for zero and nonzero frequency detuning ν, showing the regions where 2D and 1D wave structures are individually or simultaneously stable. One sees that, either on increasing ε at fixed field intensity, or on increasing the field intensity at fixed ε, one crosses a region where the only stable structure corresponds to hexagonal wave patterns.

Hence we have here another example where the stabilization of a pattern by an external field of a different symmetry is made possible by the nonvariational character of the dynamics. As for the 2ω forcing discussed



curve 1 : stability limit of 2D wave pattern
curve 2 : stability limit of 1D traveling wave

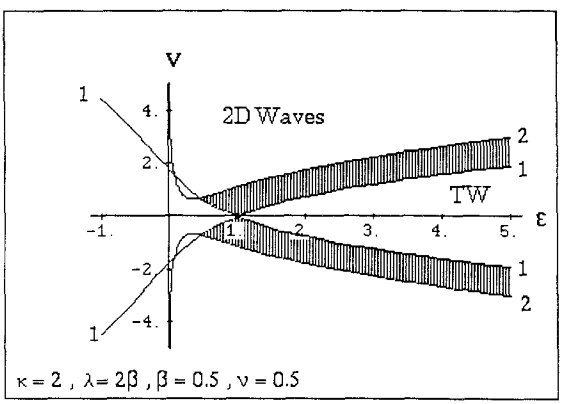


Fig. 2. Phase diagram for a Hopf bifurcation forced by temporal modulations of frequency close to three times the critical frequency, at zero and nonzero detuning.

above, this phenomenon should be experimentally observable in binary fluid convection or in the electrohydrodynamic instability of nematics. In the latter case, however, specific effects related to the intrinsic anisotropy of the system should occur.

Effectively, as discussed above, we write the uniform amplitude equations for traveling waves of wavevector $\mathbf{q}$ making an angle ϕ with the easy axis $0x$ as

$$\begin{aligned}\dot{A} &= [\varepsilon - \xi_0^2(q^2 - q_c^2)^2 - \rho(q, \phi)]A - (1 + i\beta)A|A|^2 - (\gamma + i\delta)A|B|^2 \\ \dot{B} &= [\varepsilon - \xi_0^2(q^2 - q_c^2)^2 - \rho(q, \phi)]B - (1 - i\beta)B|B|^2 - (\gamma - i\delta)B|A|^2\end{aligned} \tag{12}$$

where

$$\rho(q, \phi) = \rho q^2 \sin^2\phi + q^4(2\eta \sin^2\phi \cos^2\phi + \tau \sin^4\phi)$$

and the selected structure still corresponds to traveling waves with wavevectors $|\mathbf{q}| = q_c$ parallel to the easy axis. Of course, these equations are only valid in the weak anisotropy limit, i.e., when the anisotropy coefficients ρ, η, and τ scale as ε. Nevertheless, as in other systems, one may expect that this description remains valid when these coefficients and ε are of order one.

The application of a temporal modulation of frequency close to $3\omega_c$ still induces strong resonances between these waves of amplitude A_1 and waves traveling in directions making $2\pi/3$ angles with the easy axis and of amplitude A_2 and A_3. On assuming for simplicity that the frequency is not affected by the anisotropy, the corresponding uniform amplitude equations are

$$\begin{aligned}\dot{A}_1 &= (\varepsilon + iv)A_1 + v\bar{A}_2\bar{A}_3 - (1 + i\beta)A_1|A_1|^2 - (\kappa + i\lambda)A_1(|A_2|^2 + |A_3|^2) \\ \dot{A}_2 &= (\varepsilon - \bar{\rho} + iv)A_2 + v\bar{A}_1\bar{A}_3 - (1 + i\beta)A_2|A_2|^2 - (\kappa + i\lambda)A_2(|A_1|^2 + |A_3|^2) \\ \dot{A}_3 &= (\varepsilon - \bar{\rho} + iv)A_3 + v\bar{A}_2\bar{A}_1 - (1 + i\beta)A_3|A_3|^2 - (\kappa + i\lambda)A_3(|A_1|^2 + |A_2|^2)\end{aligned} \tag{13}$$

As a result, travelling waves in the x direction are unstable when

$$\varepsilon_- < \varepsilon < \varepsilon_+$$

with $[\bar{\rho} = \rho(q_c, 2\pi/3)]$

$$\begin{aligned}\varepsilon_\pm = {} & (v^2 - 2\lambda v - 2\bar{\rho}(\kappa - 1) \pm \{v^4 - v^2[4\lambda v + 6\bar{\rho}(\kappa - 1)] - 4[v(\kappa - 1) - \bar{\rho}\lambda]^2\}^{1/2}) \\ & \times \{2[\lambda^2 + (\kappa - 1)^2]\}^{-1}\end{aligned} \tag{14}$$

Hence, in this regime, 2D wave patterns are expected, which correspond to the fixed points of Eq. (13). However, since the relative phases of the underlying modes obey the equation

$$\partial_t(\phi_1 - \phi_2) = \partial_t(\phi_1 - \phi_3) = \Omega = (R_1^2 - R_2^2)\left(\lambda - \beta - \frac{v}{R_1}\right)\sin\left(\sum_{i=1}^{3} \phi_i\right) \tag{15}$$

even in the presence of a stable fixed point for the amplitudes R_i and the global phase $\Psi = \sum_{i=1}^{3} \phi_i$, the relative phases are time-dependent, and the resulting pattern

$$R_1 \cos(q_c x - \omega t + \phi_0) + 2R_2 \cos\left[\frac{q_c x}{2} + (\omega - \Omega)t + \phi_0\right] \cos\left(\frac{\sqrt{3}\, q_c y)}{2}\right), \quad \omega = \omega_0 - \frac{2\Omega}{2} \tag{16}$$

will in general be nonperiodic. In fact, as shown in Fig. 3, it can be viewed as an alternating sequence, periodic in the y direction, of left and right traveling waves.

Defects may play an important role in this case also. A typical example of codimension-1 defect is related to the invariance of the problem with respect to the $A_1 \to A_1$, $A_2 \to -A_2$, $A_3 \to -A_3$ symmetry. Effectively, domains of wave patterns which are out of phase by a factor π in the direction orthogonal to the easy axis may coexist in the system. As illustrated in Fig. 4, the domain wall separating two such regions corresponds to wave traveling in the x direction with a wavenumber q_c abd a frequency ω_0. The width of the defect core is of the order of the correlation length of the amplitude of the modes 2 and 3. As in other systems where transitions occur between patterns of different symmetries, such defects may be expected to trigger the transition from two-dimensional to one-dimensional wave patterns through the growth of the defect core.[25]

4. CONCLUSION

External spatial or temporal modulations may be strongly coupled with the unstable modes associated with pattern-forming instabilities. These resonances may modify the character of the bifurcation and modify the selection and stability properties of the resulting patterns. These effects were discussed in the framework of amplitude equations of the Ginzburg–Landau type. In particular, in the case of the nonvariational dynamics associated with spatiotemporal Hopf bifurcations, it was shown how pure temporal modulations may stabilize standing waves or two-dimensional wave patterns in regimes where they are otherwise unstable. These phenomena should be experimentally observable in binary fluid convection

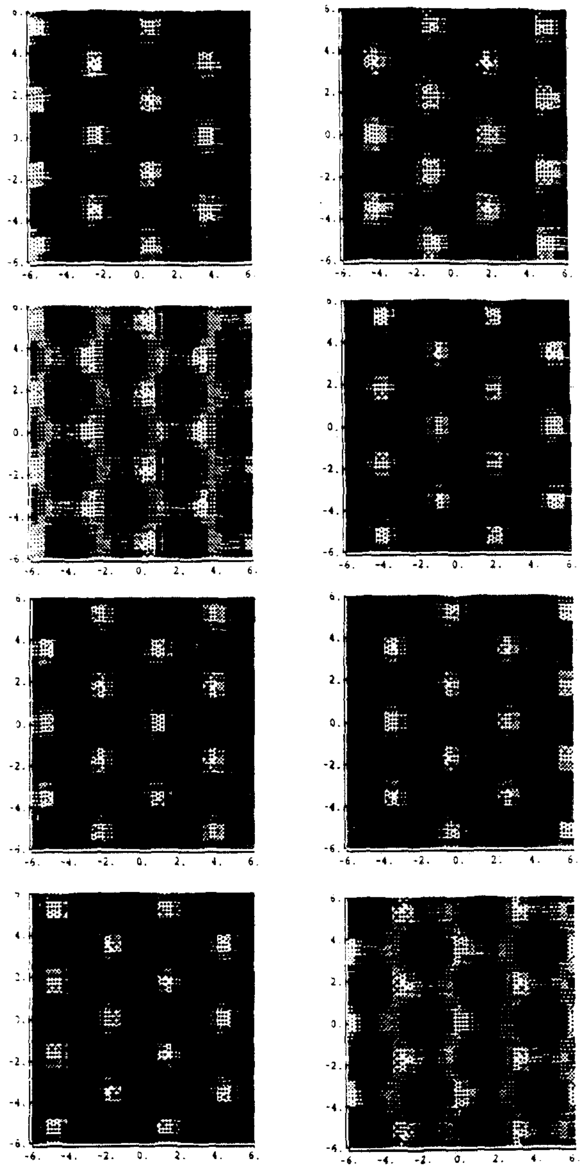


Fig. 3. Time evolution of the spatial reconstruction of the wave pattern defined by Eq. (16) ($\varepsilon = 1$, $\nu = 1$, $\beta = 1$, $\kappa = \lambda = 2$, $\alpha = 0.925$, $v = 2.82$; the time separation between two figures is $2\pi/10\omega$, and they are presented in successive order in each column, the left one corresponding to the first half period and the right one the second half period).

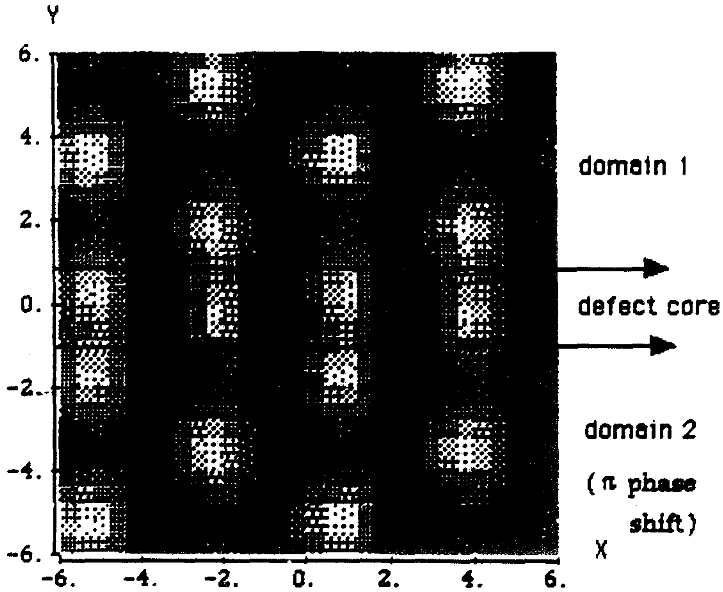


Fig. 4. Codimension-one defect separating two domains of forced wave pattern with a phase shift of π in the y direction.

or in liquid crystal instabilities such as the electrohydrodynamic instability of nematics. In the liquid crystal case, however, one has to take into account the intrinsic anisotropy of the system. For example, in the case, temporal modulations of frequencies close to three times the critical frequency should be able to induce 2D wave patterns of triangular symmetry but nonperiodic in time.

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